Linear Equations: Solutions Using Matrices with Three Variables

Solving a system of equations by using matrices is merely an organized manner of using the elimination method.

Example 1

Solve this system of equations by using matrices.

$$\begin{cases} 2x+y-3z=-4 \\ 4x-2y+z=9 \\ 3x+5y-2z=5 \end{cases}$$

The goal is to arrive at a matrix of the following form.

$$\left[\begin{array}{ccc:c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right]$$

To do this, you use row multiplications, row additions, or row switching, as shown in the following.

Put the equation in matrix form.

$$\left.\begin{array}{l} 2x+y-3z=-4 \\ 4x-2y+z=9 \\ 3x+5y-2z=5 \end{array}\right\} \rightarrow \left[\begin{array}{ccc:c} 2 & 1 & -3 & -4 \\ 4 & -2 & 1 & 9 \\ 3 & 5 & -2 & 5 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Eliminate the *x*-coefficient below row 1.

$$\begin{array}{rl} \text{Retain row } (1) & \left[\begin{array}{ccc:c} 2 & 1 & -3 & -4 \\ 0 & -4 & 7 & 17 \\ 0 & 7 & 5 & 22 \end{array}\right] \\ \text{Add } -2 \text{ times row } (1) \text{ plus 1 times row } (2) & \\ \text{Add } -3 \text{ times row } (1) \text{ plus 2 times row } (3) & \end{array} \begin{array}{l} (4) \\ (5) \\ (6) \end{array}$$

Eliminate the *y*-coefficient below row 5.

$$\begin{array}{rl} \text{Retain row } (4) & \left[\begin{array}{ccc:c} 2 & 1 & -3 & -4 \\ 0 & -4 & 7 & 17 \\ 0 & 0 & 69 & 207 \end{array}\right] \\ \text{Retain row } (5) & \\ \text{Add 7 times row } (5) \text{ plus 4 times row } (6) & \end{array} \begin{array}{l} (7) \\ (8) \\ (9) \end{array}$$

Reinserting the variables, this system is now

$$\begin{aligned} 2x + y - 3z &= -4 \qquad (7) \\ -4y + 7z &= 17 \qquad (8) \\ 69z &= 207 \qquad (9) \end{aligned}$$

Equation (9) now can be solved for *z*. That result is substituted into equation (8), which is then solved for *y*. The values for *z* and *y* then are substituted into equation (7), which then is solved for *x*.

$$
\begin{aligned}
69z &= 2 \\
z &= 3 \\
-4y + 7z &= 17 \\
-4y + 7(3) &= 17 \\
-4y - 21 &= 17 \\
-4y &= -4 \\
y &= 1 \\
2x + y + 3z &= -4 \\
2x + 1 - 3(3) &= -4 \\
2x + 1 - 9 &= -4 \\
2x &= 4 \\
x &= 2
\end{aligned}
$$

The check is left to you. The solution is $x = 2$, $y = 1$, $z = 3$.

Example 2

Solve the following system of equations, using matrices.

$$
\begin{cases}
4x + 9y = 8 \\
8x + 6z = -1 \\
6y + 6z = -1
\end{cases}
$$

Put the equations in matrix form.

$$
\left.\begin{aligned}
4x + 9y &= 8 \\
8x + 6z &= -1 \\
6y + 6z &= -1
\end{aligned}\right\} \rightarrow
\left[\begin{array}{ccc:c}
4 & 9 & 0 & 8 \\
8 & 0 & 6 & -1 \\
0 & 6 & 6 & -1
\end{array}\right]
\begin{matrix}
(1) \\ (2) \\ (3)
\end{matrix}
$$

Eliminate the x-coefficient below row 1.

$$\begin{array}{rl} \text{Retain row } (1) & \left[\begin{array}{cccc} 4 & 9 & 0 & 8 \\ 0 & 6 & 6 & -1 \\ 0 & -18 & 6 & -17 \end{array}\right] \\ \text{Replace row } (2) \text{ with row } (3) & \\ \text{Add } -2 \text{ times row } (1) \text{ plus 1 times row } (2) & \end{array} \begin{array}{l} (4) \\ (5) \\ (6) \end{array}$$

Eliminate the *y*-coefficient below row 5.

$$\begin{array}{rl} \text{Retain row } (4) & \left[\begin{array}{cccc} 4 & 9 & 0 & 8 \\ 0 & 6 & 6 & -1 \\ 0 & 0 & 24 & -20 \end{array}\right] \\ \text{Retain row } (5) & \\ \text{Add 3 times row } (5) \text{ plus 1 times row } (6) & \end{array} \begin{array}{l} (7) \\ (8) \\ (9) \end{array}$$

Reinserting the variables, the system is now:

$$4x + 9y = 8 \qquad (7)$$

$$6y + 6z = -1 \qquad (8)$$

$$24z = -20 \qquad (9)$$

Equation (9) can be solved for *z.*

$$24z = -20$$

$$z = -\frac{20}{24}$$

$$z = -\frac{5}{6}$$

Substitute $z = -\frac{5}{6}$ into equation (8) and solve for *y*.

$$6y + 6z = -1$$

$$6y + 6\left(-\frac{5}{6}\right) = -1$$

$$6y - 5 = -1$$

$$6y = 4$$

$$y = \frac{4}{6}$$

$$y = \frac{2}{3}$$

Substitute $y=\frac{2}{3}$ into equation (7) and solve for *x*.

$$\begin{aligned} 4x+9y&=8\\ 4x+9\left(\frac{2}{3}\right)&=8\\ 4x+6&=8\\ 4x&=2\\ x&=\frac{2}{4}\\ x&=\frac{1}{2} \end{aligned}$$

The check of the solution is left to you. The solution is $x=\frac{1}{2},\ y=\frac{2}{3},\ z=-\frac{5}{6}$.